ISA model: A constitutive model for soils with yield surface in the intergranular strain space

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SUMMARY

In this article, a new constitutive model for soils is proposed. It is formulated by means of plasticity, but in contrast to the precedent works, it presents a yield function describing a surface within the intergranular strain space. This latter is a state variable providing information of the recent strain history. An expression for the plastic strain rate has been proposed to guarantee the stress rate continuity. Under the application of medium or large strain amplitudes, the constitutive equation becomes independent of the intergranular strain and delivers a mathematical structure similar to some Karlsruhe hypoplastic models. Some simulations of monotonic and cyclic triaxial test are provided to evaluate and analyze the model performance. Copyright © 2015 John Wiley & Sons, Ltd.

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1. INTRODUCTION

In the last decades, several theories have been developed to capture the mechanical behavior of the soils. The most popular approaches are usually ‘stress-based’ models; that is, the model presents a single or multiple yield surfaces within the stress space. Among them, one can find conventional critical state models [1, 2], bounding surface models [3, 4], multisurface plasticity models [5–7], generalized plasticity [8, 9], or subloading plasticity models [10–12]. Although they have shown to be very competent, some limitations are often observed especially under cyclic loading: models with very narrow yield surface (e.g., [3, 4, 13]) are not able to ‘remember’ the stress history of the material as shown by the experiments [14, 15] or might present overshooting when incorporating ‘loading-initialization’ tensors. In contrast, models with greater yield surface (as in [16]) do not catch the observed plastic accumulation within this surface or cannot capture accurately the stiffness degradation under cyclic loading with small strain amplitudes (\( \| \Delta \varepsilon \| < 1 \times 10^{-3} \)) [17, 18].

These and other shortcomings allowed some alternative frameworks to emerge. Examples of them are the hyperplastic models [19–21], the Karlsruhe hypoplastic models [22–24] extended with intergranular strain [25, 26], endochronic models [27, 28], and other models defining the yield surface within the strain space [29–32]. All these frameworks are said to be strain based because they do not require a yield condition describing a surface within the stress space. By doing this, they have shown that some of the mentioned limitations are addressed and a more realistic description of the elastic locus is in many cases provided. Moreover, these models seem to avoid complicated integration algorithms considering that their formulation is compatible with most of the finite element codes whereby a strain increment is given as input.
Among these alternative models, the Karlsruhe hypoplasticity has undoubtedly shown to be a competent framework for the description of the mechanical behavior of soils. Its main idea behind is to explain the behavior of the material through a single tensorial equation linking the stress tensor rate $\dot{\sigma}$ with the strain tensor rate $\dot{\varepsilon}$. Analysis and review of this framework can be found in [26, 33–35]. Since the pioneer work of Kolymbas [22, 23], many versions of hypoplastic models as in [24, 36–38] proposed constitutive equations keeping a similar mathematical structure. Most of them can be rewritten under the following general equation [26]:

$$\dot{\sigma} = \dot{\sigma}(\sigma, \varepsilon, \dot{\varepsilon}) = E : (\dot{\varepsilon} - \dot{\varepsilon} m)$$

whereby $E = E(\sigma, \varepsilon)$ is the stiffness tensor, $Y = Y(\sigma, \varepsilon)$ is a scalar function named the ‘degree of non-linearity’ [26], $m = m(\sigma, \varepsilon)$ is the hypoplastic flow rule tensor [26, 33], and $\varepsilon$ is the void ratio. Under this mathematical structure, the conventional strain decomposition into an elastic and plastic component, as, for example, in elastoplastic models, is not possible despite some authors having deduced some expressions to compute the accumulated (plastic) strain after an infinitesimal stress loop [26, 33]. The experience with Karlsruhe hypoplastic models following Equation 1 suggests that good simulation capabilities are delivered only under medium and large strain amplitudes $\| \Delta \varepsilon \| > 10^{-3}$. In contrast, simulations of small strain effects, such as the increase of the stiffness upon reversal loading or material memory upon re-loading paths, are very unlikely. This drawback motivated some subsequent versions to desert from the ‘single tensorial equation’ idea. Niemunis and Herle [25], for example, proposed the so-called ‘hypoplasticity with intergranular strain’ to improve simulations of cyclic loading under small strain amplitudes $\| \Delta \varepsilon \| < 10^{-3}$. Their model was expressed through two equations and incorporated the intergranular strain as an additional state variable [25, 26]. Doing this, they achieved to simulate the stiffness increase upon reversal loading but failed to reproduce the memory effects upon reloading paths. The latter shortcoming was actually related with the overshooting observed in some simulations. Long after, Fuentes et al. [39] proposed the so-called ‘hypoplastic model with loading surface’ aiming to reproduce the observed memory effects upon reloading paths. The achievement in this direction was clear, but the increase of stiffness due to reversal loading was not properly reproduced. Hence, one can certainly affirm that the models of Niemunis and Herle [25] and Fuentes et al. [39] were complementing each other on their advantages and disadvantages, while a formulation capable to simulate all the mentioned small strain effects is still missing. This fact motivates to keep looking towards new mathematical approaches aiming for improved simulations under a wide range of strain amplitudes.

In this article, a novel strain-based framework for constitutive modeling is proposed. It is named intergranular strain anisotropy (ISA) model and is based on the intergranular strain concept by Niemunis and Herle [25], but contrary to this formulation, a yield function has been proposed describing a surface within the intergranular strain space. Several salient features are obtained with this, among them the existence of an elastic locus related with a strain amplitude, which enables the simulation of memory effects and the simulation of the stiffness increase upon reversal loading. The model is elastoplastic but as explained in the following sections, its formulation delivers a similar equation to some Karlsruhe hypoplastic models (Equation 1) for medium and large strain amplitudes $\| \Delta \varepsilon \| > 10^{-3}$. This article is organized as follows: the first section is dedicated to describe the elastoplastic evolution equation of the intergranular strain $\mathbf{h}$. Subsequently, the formulation of the constitutive model is given. The material parameters are then described and a short guide for their calibration is provided. At the end, some aspects of the numerical integration are explained, and some simulations with monotonic and cyclic triaxial tests are given to evaluate the proposed model.

The notation of this article is as follows. Scalar quantities are denoted with italic fonts (e.g., $a$ and $b$), second rank tensors with bold fonts (e.g., $\mathbf{A}$ and $\mathbf{\sigma}$), and fourth rank tensors with Sans Serif type (e.g., $E$ and $L$). Multiplication with two dummy indices, also known as double contraction, is denoted with a colon ‘:’ (e.g., $\mathbf{A} : \mathbf{B} = A_{ij} B_{ij}$). When the symbol is omitted, it is then interpreted as a dyadic product (e.g., $\mathbf{AB} = A_{ij} B_{kl}$). The deviatoric component of a tensor is symbolized with an asterisk as superscript $\mathbf{A}^*$. The effective stress tensor is denoted with $\mathbf{\sigma}$ and the strain tensor with $\mathbf{\varepsilon}$. The Roscoe invariants are defined as $p = -tr\mathbf{\sigma}/3$, $q = \sqrt{3}/2 \| \mathbf{\sigma}^* \|$, $\varepsilon_v = -tr\mathbf{\varepsilon}$, and $\varepsilon_s = \sqrt{2}/3 \| \mathbf{\varepsilon}^* \|$. The stress ratio $\eta$ is defined as $\eta = q/p$. 

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2. INTERGRANULAR STRAIN MODEL

The intergranular strain tensor \( \mathbf{h} \) is a strain-type state variable providing information about the recent strain history of the material and was firstly proposed by Niemunis and Herle [25]. In this work, a different approach to describe the evolution of the intergranular strain is proposed aiming to improve the simulations of the memory effects upon reloading paths. For this purpose, the evolution equation relating the rate of the intergranular strain \( \dot{\mathbf{h}} \) with the rate of the strain tensor \( \dot{\mathbf{e}} \) is proposed by means of the elastoplastic relation:

\[
\dot{\mathbf{h}} = \dot{\mathbf{e}} - \dot{\lambda}_H \mathbf{N}
\]  

whereby \( \dot{\lambda}_H \) is the consistency parameter \( \dot{\lambda}_H \geq 0 \) related with the yield function \( F_H \) and \( \mathbf{N} \) is the intergranular strain flow rule. The yield function \( F_H = F_H(\mathbf{h}, \mathbf{c}) = 0 \) depends on the intergranular strain \( \mathbf{h} \) and the hardening variable \( \mathbf{c} \). The formulation of the yield function \( F_H \) is very simple: it describes a (hyper-)sphere within the intergranular strain space through the relation:

\[
\text{IS yield surface: } F_H = \| \mathbf{h} \| - R/2 = 0
\]

whereby tensor \( \mathbf{c} \) describes the center of the sphere and is termed ‘back-intergranular strain’ and \( R \) is a material parameter representing the diameter of the yield surface. In the space of the volumetric invariant \( h_v/\sqrt{3} = -\text{tr} (\mathbf{h})/\sqrt{3} \) and the deviator invariant \( \sqrt{3}/2h_* = \| \mathbf{h}^* \| \) where \( \mathbf{h}^* \) is the deviator intergranular strain, the yield surface from Equation 3 takes exactly the form of a circle, as illustrated in Figure 1(a).

For the sake of simplicity, the flow rule tensor \( \mathbf{N} \) is proposed to be unit \( \| \mathbf{N} \| = 1 \) and normal to the yield surface (Figure 1(a)):

\[
\mathbf{N} = (\mathbf{h} - \mathbf{c})/(R/2) = (\mathbf{h} - \mathbf{c})^{\rightarrow}
\]

whereby the arrow implies normalization of a tensor, for example, \( (\mathbf{A})^{\rightarrow} = \mathbf{A}/\| \mathbf{A} \| \). Under elastic conditions \( F_H < 0 \), the consistency parameter takes \( \dot{\lambda}_H = 0 \), and Equation 2 yields

\[
\dot{\mathbf{h}} = \dot{\mathbf{e}} \quad \text{for} \quad F_H < 0
\]

The last equation in conjunction with the yield function (Equation 3) suggests that the model delivers an elastic response only for very small strain amplitudes \( \| \Delta \mathbf{e} \| < R \). On the other hand, for the plastic condition \( F_H = 0 \) and \( \dot{\lambda}_H \geq 0 \), a hardening equation for \( \mathbf{c} \) is required. For this purpose, some concepts of the bounding surface plasticity are borrowed [4]. A bounding surface within the intergranular strain space has been introduced as depicted in Figure 1(a). The proposed bounding
The bounding surface function is

\[ F_{Hb} \equiv \| \mathbf{h} \| - R = 0 \]  

(6)

The evolution equation for tensor \( \mathbf{c} \) describes its hardening mechanism under plastic conditions. The general form of the hardening equation is

\[ \mathbf{c} = \lambda_H \mathbf{c} \]  

(7)

whereby \( \mathbf{c} \) is the hardening function of tensor \( \mathbf{c} \). Of course, under elastic conditions, no hardening is experienced, \( \mathbf{c} = 0 \), because \( \lambda_H = 0 \). Under plastic conditions, its hardening depends on the proposed function for \( \lambda_H \). This follows from the following observations: willing to fulfill the bounding surface constraint \( \| \mathbf{h} \| \leq R \), the maximum magnitude of \( \mathbf{c} \) should be therefore bounded with \( \| \mathbf{c} \| = R/2 \), as illustrated in Figure 1(b). It is desired that tensor \( \mathbf{c} \) evolves towards an ‘image’ tensor at the bounding surface denoted by \( \mathbf{c}_b \) and defined as

\[ \mathbf{c}_b = R/2(\mathbf{c}) \rightarrow \]  

(8)

Notice the mapping rule using the strain rate direction in Equation 8. Considering this, a very simple hardening function \( \lambda_H \) is proposed as follows:

\[ \mathbf{c} = \beta (\mathbf{c}_b - \mathbf{c})/R \]  

for \( F_H = 0 \)  

(9)

whereby \( \beta \) is a material parameter controlling its hardening rate.

The solution of the consistency parameter \( \lambda_H \) requires the conventional loading–unloading relations for elastoplastic formulations. At the yield surface \( F_H = 0 \), the consistency condition \( \dot{F}_H = (\partial F_H/\partial h) : \dot{h} + (\partial F_H/\partial c) : \dot{c} = 0 \) holds. Replacing Equations 2, 3, 7, and 9 in \( \dot{F}_H = 0 \) and using the derivatives

\[ \frac{\partial F_H}{\partial h} = \mathbf{N} \]  

(10)

\[ \frac{\partial F_H}{\partial c} = -\mathbf{N} \]  

(11)

yields to the definition of the consistency parameter \( \lambda_H \):

\[ \lambda_H = \frac{\langle \mathbf{N} : \dot{\mathbf{c}} \rangle}{1 + H_H} \]  

(12)

where the operator \( \langle \rangle \) is the Macaulay brackets (\( \langle \| \rangle = \| \) when \( \| > 0 \) and \( = 0 \) when \( \| \leq 0 \)) and \( H_H = -(\partial F_H/\partial c) : \dot{c} \) is the hardening modulus.

The presented relations complete the set of equations of the intergranular strain model. From the plasticity perspective, the model is quite simple because it presents an associated flow rule \( \mathbf{N} = \partial F_H/\partial \mathbf{h} \), a single and simple hardening mechanism (Equation 9), and an identity elastic tensor (Equation 5). In the following lines, a scalar function is introduced to quantify how close is the intergranular strain \( \mathbf{h} \) from the bounding surface \( F_{Hb} = 0 \).

Similarly to the image tensor \( \mathbf{c}_b \) defined in Equation 8, one can propose an image tensor of the intergranular strain at the bounding surface denoted by \( \mathbf{h}_b \) and defined as

\[ \mathbf{h}_b = R\mathbf{N} \]  

(13)

Notice the mapping rule in Equation 13 with the intergranular strain flow rule \( \mathbf{N} \) previously defined in Equation 4. The distance \( \| \mathbf{h}_b - \mathbf{h} \| \) provides information of how close is the intergranular strain \( \mathbf{h} \) to the bounding surface \( F_{Hb} = 0 \). According to the proposed model, the bounding condition \( \mathbf{h} = \mathbf{h}_b \) should be asymptotically reached after applying large strains in a constant direction \( \mathbf{c} \). This particular state \( \mathbf{h} = R\mathbf{c}/2 \) is called the ‘fully mobilized’ state. For the sake of comparison, the scalar function \( \rho \) is proposed:

\[ \rho = 1 - \frac{\| \mathbf{h}_b - \mathbf{h} \|}{2R} \]  

(14)
This scalar function renders \( \rho = 0 \) when \( \| h_b - h \| = 2R \) (strain reversal after fully mobilized state) and \( \rho = 1 \) when \( h = h_b \) (fully mobilized state) and will be used in the next section for the formulation of the mechanical model.

3. MECHANICAL MODEL FORMULATION

The constitutive model for the mechanical behavior relates the rate of the (effective) stress tensor \( \dot{\sigma} \) with the rate of the strain tensor \( \dot{\epsilon} \) and is dictated by the elastoplastic relation:

\[
\dot{\sigma} = E : \left( \dot{\epsilon} - \dot{\epsilon}^P \right) \tag{15}
\]

where \( E \) is the stiffness tensor and \( \dot{\epsilon}^P \) is the plastic strain rate tensor defined as

\[
\dot{\epsilon}^P = \| \dot{\epsilon}^P \| m
\]

with the flow rule tensor \( m \). The formulation of the magnitude \( \| \dot{\epsilon}^P \| \) must be consistent with the elastoplastic formulation of the intergranular strain model, meaning that when the intergranular strain \( h \) lies within the elastic locus \( F_H < 0 \), an elastic response of the mechanical constitutive model (Equation 15) is to be expected, that is, \( \dot{\sigma} = E : \dot{\epsilon} \). With the same reasoning, a plastic step in the intergranular strain space \( F_H = 0 \) would produce a plastic response of the mechanical model (Equation 15). This requirement can be fulfilled very easily by proposing the magnitude \( \| \dot{\epsilon}^P \| \) to be proportional to the consistency parameter \( \lambda_H \sim \langle N : \dot{\epsilon} \rangle \) already defined in Equation 12. A suitable relation is

\[
\| \dot{\epsilon}^P \| = Y \rho^{x_h} \langle N : \dot{\epsilon} \rangle \tag{17}
\]

where \( x_h \) is a parameter to control the reduction of the plastic strain rate during unloading paths and \( Y \) is the value of \( \| \dot{\epsilon}^P \| \) at the fully mobilized state (\( \rho = 1 \)). Now, the same definition of \( \| \dot{\epsilon}^P \| \) can be rewritten in the following way:

\[
\| \dot{\epsilon}^P \| = y_h Y \| \dot{\epsilon} \|, \quad \text{and} \quad y_h = \rho^{x_h} \langle N : \dot{\epsilon} \rangle \tag{18}
\]

whereby the factor \( y_h \) is the scalar function depending on the intergranular strain \( h \) and responsible of the reduction of \( \| \dot{\epsilon}^P \| \) due to an unloading process. If \( y_h = 0 \), the response is elastic, whereas \( y_h = 1 \) implies fully mobilized states. It is reminded that according to the intergranular strain model, fully mobilized states (\( \rho = 1 \) or \( y_h = 1 \)) is only reached after the application of large strains in a constant direction. Substitution of the condition \( y_h = 1 \) in the proposed constitutive model (Equations 15, 16, and 18) leads to a mathematical structure similar to some Karlsruhe hypoplastic models (Section 1):

\[
\dot{\sigma} = E : (\dot{\epsilon} - Y m \| \dot{\epsilon} \|) \quad \text{for} \quad y_h = 1 \tag{19}
\]

whereby the factor \( Y \) corresponds to the ‘degree of non-linearity’ [26], name hereafter adopted. This property makes the proposed model very attractive considering the fact that for many authors, the Karlsruhe hypoplastic models simulate very well the material behavior at medium and large strain amplitudes (\( \| \Delta \epsilon \| > 10^{-3} \)). For cyclic loading, the proposed model considers the small strain effects (stiffness increase, reduction of plastic strain rate) through the factor \( 0 \leq y_h \leq 1 \) (Equation 18). Furthermore, a ‘memory’ is provided through the IS yield surface, which in turn eliminates the overshooting. The proposed constitutive relations allow to formulate different models under the same ISA framework. It requires the definition of the elastic stiffness tensor \( E \), the flow rule \( m \), and the degree of non-linearity \( Y \). In the next sections, a model under this mathematical framework will be proposed. The definitions of \( E, m, \) and \( Y \) will be based on the characteristic void ratios (maximum and critical void ratio) and the characteristic stress surfaces (bounding, dilatancy, and critical stress surface) defined in the sequel.
3.1. Characteristic void ratios

The model introduces two pressure-dependent characteristic void ratios, the maximum void ratio at isotropic compression \( e_i = e_i(p) \) and the critical state void ratio \( e_c = e_c(p) \) whereby \( p = -1/3\, \text{tr}\, \sigma \) is the mean pressure. Both are in general refereed as the characteristic void ratios. The maximum void ratio \( e_i \) can be obtained with the integration of the differential equation \( \dot{p} = K^L \dot{\varepsilon}_v \), whereby \( K^L \) is the bulk modulus at isotropic loading and \( \dot{\varepsilon}_v = -\dot{\varepsilon} \) is the volumetric strain rate. For the proposed model, the following function is adopted for \( K^L \):

\[
K^L = \frac{1}{\lambda_i} p^{n_{pi}} \left( 1 + \frac{e}{e_n} \right)
\]

(20)

where \( \lambda_i > 0 \), \( n_{pi} > 0 \) and \( e_n > 1 \) are material parameters. The integration of \( \dot{p} = K^L \dot{\varepsilon}_v \) yields to the expression:

\[
e_i = \left[ \frac{-\lambda_i p^{1-n_{pi}} (1 - n_e)/(1 - n_{pi}) + e_i^{1-n_{pi}}}{1/(1-n_e)} \right]^{1/(1-n_e)}
\]

(21)

where \( e_{i0} \) is the value of the maximum void ratio \( e_i \) at \( p = 0 \). The critical void ratio \( e_c \) follows a similar relation as the one from [37]:

\[
e_c = e_{c0} \exp \left( \lambda_c (p^{1-n_{pc}})/(n_{pc} - 1) \right)
\]

(22)

where \( e_{c0} \) is the critical void ratio \( e_c \) at \( p = 0 \) and \( \lambda_c \) and \( n_{pc} \) are material parameters.

3.2. Characteristic stress surfaces and mapping rules

The model considers three characteristic surfaces defined within the stress space: the critical state surface, the dilatancy surface, and the bounding surface. These concepts have been already employed in existing models, for example, [39–43], and operate in the proposed model as well.

According to the cited models, the definition of these stress surfaces is accompanied with the introduction of a mapping rule. The purpose of the mapping rule is to project at these surfaces an image of the current stress ratio tensor \( r = \sigma^* / p \), whereby \( \sigma^* \) is the deviator stress tensor. The direction of the projection is defined by the traceless tensor \( n \) named the deviatoric loading tensor. Analysis of different approaches to define the deviatoric tensor \( n \) can be found in [13]. The simplest mapping rule would be to set \( n = \mathbf{r} \) with the known impediment of not distinguishing unloading paths whereby eventually \( \dot{\vartheta} = \mathbf{r} : \mathbf{N}^* < 0 \). To overcome this issue, the following mapping rule is proposed:

\[
n = \begin{cases} \mathbf{r}, & \text{for } \dot{\vartheta} > 0 \\ \mathbf{r} - \dot{\vartheta} \left( \mathbf{N}^* - \mathbf{r} \right), & \text{for } \dot{\vartheta} \leq 0 \end{cases}
\]

(23)

with \( \dot{\vartheta} = \mathbf{r} : \mathbf{N}^* \)

This mapping rule is simple because it distinguishes between loading and unloading paths and does not introduce loading-initialization tensors as in [43]. Moreover, under monotonic loading, it trends to \( n = \mathbf{r} \), which in turn gives numerical stability and eliminates the intergranular strain dependence in the model (\( \mathbf{r} = \mathbf{r}(e, \sigma, \dot{e}) \)). Having defined the deviatoric loading tensor \( n \), it is now proceeded with the definition of the characteristic stress surfaces. The first corresponds to the critical state surface described with the function:

\[
F_c \equiv \mathbf{r} : \mathbf{n} - r_c = 0, \quad r_c = \sqrt{2/3} M_e g(\theta_n)
\]

(24)

where \( M_e \) is the critical state slope for triaxial extension and the scalar function \( g = g(\theta_n) \) is a factor evaluated with the Lode’s angle \( \theta_n \) of the deviatoric loading tensor \( n \). The function \( g = g(\theta_n) \) takes values within the range \( c \leq g \leq 1 \), whereby the factor \( c = M_e / M_c \) represents the ratio between the critical state slope for triaxial extension \( M_e \) and triaxial compression \( M_c \). For the scalar function \( g \), the following relation is adopted:

\[
g(\theta) = \frac{2c}{(1 + c) - (1 - c) \cos(3\theta)}
\]

(25)
The dilatancy surface is the one at which the volumetric plastic strain rate \( \dot{\varepsilon}_v^P = -\text{tr}\dot{\varepsilon}_v^P \) changes sign, for example, from plastic compressive behavior \( (\dot{\varepsilon}_v^P > 0) \) to plastic dilative behavior \( (\dot{\varepsilon}_v^P < 0) \). For the current model, the function proposed by Dafalias [41] is adopted:

\[
\text{dilatancy surface: } F_d = r : n - r_c f_d = 0, \quad f_d = \exp(n_d (e - e_c)) \tag{26}
\]

where \( n_d > 0 \) is a material parameter and \( e_c = e_c(p) \) is the critical state void ratio previously defined in Equation 22.

The bounding surface is the one at which the stress rate vanishes \( \dot{\sigma} = 0 \) when \( \overrightarrow{e} = m \). This definition is similar to some hypoplastic models [26, 44] but differs from some bounding surface models (e.g., [45]) for which at the bounding surface, the rate of the stress ratio (and not of the stress) vanishes \( \dot{r} = 0 \). For this model, a wedge-capped type bounding surface is proposed as depicted in Figure 2(b). The bounding surface intercepts the isotropic axis \( q = 0 \) when the void ratio is equal to its maximum possible \( e = e_i(p) \). The following function is suitable for this purpose:

\[
\text{bounding surface: } F_b = r : n - r_c f_b = 0, \quad f_b = f_{b0} \left( 1 - \left( \frac{e}{e_i} \right)^{n_F} \right)^{1/2} \tag{27}
\]

where \( f_{b0} \approx 1.3 \) is a parameter controlling the bounding surface slope in the \( p - q \) space at the limit \( p = 0 \) and \( n_F \) is an exponent to be defined. It is desired that the bounding surface intercepts the critical state surface \( \eta = M \) when the void ratio is equal to the critical state void ratio \( e = e_c \); see, for example, point \( B_1 \) in Figure 2. Hence, an expression for \( n_F \) can be solved by setting \( f_b = 1 \) and \( e = e_c \) in Equation 27 and gives

\[
n_F = \frac{\log \left( \frac{(f_{b0}^2 - 1)}{f_{b0}^2} \right)}{\log(e_c/e_i)} \tag{28}
\]

Figure 2(a) shows the characteristic void ratio curves \( e = e_i(p) \) and \( e = e_c(p) \), and their interceptions for the void ratios equal to \( e = 0.96 \) (see points \( A_1 \) and \( B_1 \)) and to \( e = 0.94 \) (see points \( A_2 \) and \( B_2 \)). The same four points can be also identified within the \( p - q \) space in Figure 2(b). Therein, the bounding surface \( F_b = 0 \) is plotted according to Equation 27. Notice that the interception of the bounding surface \( F_b = 0 \) with the critical state line lies exactly at the point \( B_1 \) for the void ratio \( e = 0.96 \) and at \( B_2 \) for \( e = 0.94 \).

Finally, the image of the stress ratio tensor \( r \) are projected at the critical, dilatancy, and bounding surface using the following mapping rule:
\[ r_c = r_c n, \quad \text{with} \quad r_c = \sqrt{2/3} M \]  
\[ r_d = r_d n, \quad \text{with} \quad r_d = r_c f_d \]  
\[ r_b = r_b n, \quad \text{with} \quad r_b = r_c f_b \]  
with \( M = M_c g(\theta_a) \).

### 3.3. Stiffness tensor

The formulation of the stiffness tensor \( E \) follows a similar reasoning to the one by Niemunis and Herle [25] and by Papadimitriou and Dafalias [42]. Their formulation enables to simulate the increase of the stiffness due to reversal loading as observed in many experiments performed under small strain amplitudes \( \| \Delta \varepsilon \| < 10^{-3} \). The following relation is herein adopted:

\[ E = m \tilde{E} \]  
where \( m \geq 1 \) is a scalar function responsible of the stiffness increase among reversal loading and \( \tilde{E} = \bar{E}(\sigma, e) \) is the residual elastic stiffness at fully mobilized states \( \rho = 1 \). The residual stiffness \( \tilde{E} \) depends only on the stress tensor \( \sigma \) and the void ratio \( e \).

The maximum value of \( m \) corresponds to \( m = m_R \), considered herein as a material parameter. At the elastic condition \( y_h = 0 \) (Equation 18), the factor \( m \) takes its maximum \( m = m_R \), while at the fully mobilized state \( y_h = 1 \), it takes its minimum \( m = 1 \). A suitable interpolation function is

\[ m = m_R + (1 - m_R) y_h \]  
Note that Equation 33 guarantees the stress rate continuity between the elastic and plastic response; that is, for neutral loading \( (F_H = 0, \dot{\lambda}_H = 0, \| \dot{\varepsilon} \| > 0) \), the factor recovers its maximum value \( m = m_R \) adopted also at elastic conditions. To illustrate the behavior of the scalar functions \( m \) and \( \rho^h \), consider the example depicted in Figure 3. It shows a fully volumetric unloading process from point \( A \) to point \( C \) using the parameter \( m_R = 3 \). From point \( A \) to point \( B \), the response is elastic, and therefore, the function \( m \) remains constant. Between points \( B \) and \( C \), the response is plastic, and thus, function \( m \) decreases until it reaches \( m = 1 \) at point \( C \). Note the different response for \( \lambda_h = 2 \) and \( \lambda_h = 9 \).

The residual stiffness \( \tilde{E} \) follows from similar relations to Fuentes et al. [39]:

\[ \tilde{E} = 3 \tilde{K} \mathbf{1} \mathbf{1} + 2 \tilde{G} \left( 1 - \frac{1}{\sqrt{3} M_c} \right) - \frac{\bar{K}}{\sqrt{3} M_c} \left( \mathbf{1} \mathbf{r} + \mathbf{r} \mathbf{1} \right) \]  

\[ \lambda_h = 9 \]
\[ \lambda_h = 2 \]
\[ \rho^h \]
\[ h_v / \sqrt{3} \]
\[ 2R \]
\[ \text{Bounding surface} \]
\[ \text{Yield surface} \]
\[ \text{elastic} \]
\[ \text{plastic} \]

Figure 3. (a) Diagram of the movement of the yield surface after a 180° reversal. (b) Behavior of the functions \( m \) and \( \rho^h \) from Equations 33 and 14, respectively.
which considers the anisotropic terms $-\tilde{K}/\sqrt{3M_c}(1\mathbf{r} + \mathbf{r})$ to account the stress ratio $r$ dependence. The following hypo-elastic relations for the bulk modulus $K = m\tilde{K}$ and shear modulus $G = m\tilde{G}$ are adopted:

$$
\tilde{K} = \frac{K^L}{(1 - Y_{im})} = \frac{1}{\lambda_i} \rho_n^{np} \frac{1 + e^{\eta e}(1 - Y_{im})}{1 + e}
$$

$$
\tilde{G} = K r , \text{ with } r = \frac{1 - 2\nu}{2(1 + \nu)}
$$

where $\nu$ is the Poisson ratio and $Y_{im} = 1/3$ is the value which takes the factor $Y$ at isotropic stress states ($q = 0$). The details of the formulation of tensor $\tilde{E}$ can be found in [26] and [39].

3.4. Plastic strain rate

The plastic strain rate tensor $\mathbf{\dot{e}}^P$ requires the definition of the degree of non-linearity $Y$ and the flow rule $\mathbf{m}$. The flow rule $\mathbf{m}$ is defined similar to the model by Dafalias [41]:

$$
\mathbf{m} = (-1/2(\mathbf{r}_d - \mathbf{r} : \mathbf{n})\mathbf{1} + \| \mathbf{N}^* \| \mathbf{n}) \rightarrow
$$

Notice that in contrast to the formulation by [41], the factor $\| \mathbf{N}^* \|$ has been introduced into the deviatoric component to simulate a smoother transition between the elastic and fully mobilized states.

The definition of the bounding surface requires that the stress rate vanishes $\dot{\sigma} = 0$ at $F_b = 0$ when $\mathbf{\dot{\epsilon}} = \mathbf{m}$ (Equation 27). According to the constitutive Equations 15 and 18, this requirement is fulfilled if the degree of non-linearity is equal to one, $Y = 1$, and the intergranular strain $\mathbf{h}$ is fully mobilized, $\gamma_b = 1$. Thus, the following function for $Y$ is proposed:

$$
Y = \left( \frac{\| \mathbf{r} - \mathbf{r}_0 \|}{\| \mathbf{r}_b - \mathbf{r}_0 \|} \right)^{n_Y}
$$

whereby $\mathbf{r}_0$ is an image stress ratio pointing in the direction $-\mathbf{n}$ and computed with the following mapping rule:

$$
\mathbf{r}_0 = -r_c f_{b0} g(\theta - \mathbf{n}) \mathbf{n}
$$

In Equation 39, the parameter $f_{b0}$ instead of the factor $f_b$ has been used, to assure that $(\mathbf{r} - \mathbf{r}_0) : \mathbf{n} > 0$ always holds. The exponent $n_Y$ is calibrated to control the value of $Y$ at isotropic states $q = 0$. For this state, the value of $Y = Y_{im}/g$ should be delivered, whereby the function $g$ has been introduced to consider the influence of the direction of $\mathbf{n}$. By replacing in Equation 38 with $Y = Y_{im}/g$, $\mathbf{r} = 0$, $\| \mathbf{r}_0 \| = \mathbf{r}_b = 0$, $\| \mathbf{r}_b - \mathbf{r}_0 \| = \sqrt{2/3M_c(f_{b0} + c_{f_{b0}})}$, and $\| \mathbf{r}_b - \mathbf{r}_0 \| = \sqrt{2/3M_c(f_{b0} + c_{f_{b0}})}$, the following exponent $n_Y$ is obtained:

$$
n_Y = \frac{\log(Y_{im}/g)}{\log(f_{b0}/(f_{b0} + f_{b}/c))}
$$

4. MATERIAL CONSTANTS

The ISA model requires the calibration of a set of parameters. Although most of them have already been used and explained in other models, it is worthy to give here a short guide for their determination. They require at least the conduction of three monotonic undrained triaxial tests, one cyclic undrained triaxial test, and two isotropic compression tests. Of course, the more experiments, the more precise is the calibration. The parameters are categorized into hypo-elasticity, critical state, characteristic surfaces, and intergranular strain as given in Table I. A suggested range of the parameters is also given in Table I. These ranges were suggested by the authors after simulating different sands.

The calibration of parameter $M_c$ follows from the linearization of the experimental data from triaxial compression tests lying at the critical state with the equation $q = M_c p$. The parameter $c$ can
Table I. Material constants of the proposed constitutive model.

<table>
<thead>
<tr>
<th>Description</th>
<th>Units</th>
<th>Approx. range</th>
<th>Toyoura (MT)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hypo-elasticity</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda_i$</td>
<td>[-]</td>
<td>$10^{-6} - 1$</td>
<td>0.005</td>
</tr>
<tr>
<td>$e_{i0}$</td>
<td>[-]</td>
<td>0.5 – 2</td>
<td>1.102</td>
</tr>
<tr>
<td>$n_{pi}$</td>
<td>[-]</td>
<td>0.3 – 1</td>
<td>0.67</td>
</tr>
<tr>
<td>$n_e$</td>
<td>[-]</td>
<td>1 – 5</td>
<td>2</td>
</tr>
<tr>
<td>$\nu$</td>
<td>[-]</td>
<td>0 – 0.5</td>
<td>0.22</td>
</tr>
<tr>
<td>Critical state (CS)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda_c$</td>
<td>[-]</td>
<td>$10^{-6} - 1$</td>
<td>0.000446</td>
</tr>
<tr>
<td>$e_{c0}$</td>
<td>[-]</td>
<td>0.5 – 2</td>
<td>0.93</td>
</tr>
<tr>
<td>$n_{pc}$</td>
<td>[-]</td>
<td>0.5 – 1</td>
<td>0.25</td>
</tr>
<tr>
<td>$M_c$</td>
<td>[-]</td>
<td>0.5 – 1.7</td>
<td>1.252</td>
</tr>
<tr>
<td>$c$</td>
<td>[-]</td>
<td>0.5 – 1</td>
<td>0.7</td>
</tr>
<tr>
<td>Characteristic surfaces</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n_d$</td>
<td>[-]</td>
<td>0 – 5</td>
<td>1.2</td>
</tr>
<tr>
<td>$f_{b0}$</td>
<td>[-]</td>
<td>1 – 2</td>
<td>1.25</td>
</tr>
<tr>
<td>Intergranular strain (IS)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$m_R$</td>
<td>[-]</td>
<td>$1 - 7$</td>
<td>5</td>
</tr>
<tr>
<td>$R$</td>
<td>[-]</td>
<td>$1 \times 10^{-5} - 5 \times 10^{-4}$</td>
<td>$1.4 \times 10^{-4}$</td>
</tr>
<tr>
<td>$\beta$</td>
<td>[-]</td>
<td>0 – 5</td>
<td>1.0</td>
</tr>
<tr>
<td>$\chi_h$</td>
<td>[-]</td>
<td>$1 - 10$</td>
<td>7</td>
</tr>
</tbody>
</table>

be computed with the Mohr–Coulomb relation $c = 3/(3 + M_c)$ or by fitting the experimental data from triaxial extension lying at the critical state. The relation $e = e_c(p)$ from Equation 22 describes the critical state line in the $e - p$ space and requires the determination of the parameters $\lambda_c$, $e_{c0}$, and $n_{pc}$. They can be adjusted to experimental data lying at the critical state, which is obtained after inducing very large deviator deformations $e_s > 25\%$ to the sample under triaxial conditions. Figure 4(a) and (b) shows two calibration examples of the critical void ratio curve $e_c = e_c(p)$: Figure 4(a) with the results of moist tamped samples (MT) of Toyoura sand, while Figure 4(b) with moist tamped and air pluviated samples (AP) of Karlsruhe fine sand.

The parameter $f_{b0}$ controls the maximum stress ratio $\eta_{max}$, which can be obtained by the model. The maximum stress ratio $\eta_{max}$ can be measured from triaxial data of very dense samples (relative index $I_D > 0.7$). Points lying at the maximum stress ratio $\eta_{max}$ can be fitted with the equation $f_{b0} = \eta_{max}/M_c$.

The hypo-elastic parameters $\lambda_i$, $n_{pi}$, and $n_e$, and $e_{i0}$ can be fitted as follows: one can mathematically show that under isotropic compression $q = 0$ and fully mobilized state $\rho = 1$, the constitutive model follows exactly Equation 21 (using the initial void ratio $e = e_0$ for $p = 0$). Thus, it is possible to fit directly the hypo-elastic parameters $\lambda_i$, $n_{pi}$, and $n_e$ with the experiments using Equation 21. A calibration example is given in Figure 4(c) with moist tamped samples MT of Toyoura sand. Notice that one of the isotropic compression curves corresponds to the maximum void ratio curve $e = e_1(p)$, which enables the calibration of the parameter $e = e_{i0}$ at $p = 0$. Figure 4(d) shows that the curves of the maximum void ratio $e = e_2(p)$ of Toyoura sand for two sample preparation methods (MT and AP) are different. This is the well-known inherent fabric effect [46], which is not considered by the proposed model. Some authors have proposed to capture the inherent fabric effect through the modification of some hardening functions and the location of the critical state line in the $e - p$ space [48–52]. According to [48, 53], the evolution of the fabric (fabric changes) can be neglected for small and medium strain amplitudes ($\| \Delta e \| < 0.02$). Considering that the simulations presented herein with AP samples of Toyoura sand are within this strain amplitude ($\| \Delta e \| < 0.02$); an evolving fabric formulation is herein not considered. Instead, a simple method similar to [54] has been adopted whereby the parameters $e_{i0}$ and $e_{e0}$ are multiplied by a factor to simulate the different fabrics arising from the sample preparation method. The two curves in Figure 4(d) show that $e_{i0} = 1.102$ for the MT sample, while $e_{i0} = 0.975$ for the AP sample. Thus, a factor of
Figure 4. Calibration of the characteristic void ratios: (a) and (b) critical void ratio curve \( e = e_c(p) \) of the Toyoura sand and Karlsruhe fine sand (KFS), respectively. (c) Isotropic compression curves samples. (d) Maximum void ratios for two different sample preparation methods. Data of Toyoura sand by Ishihara [46]. Data of KFS sand [47].

\[
\frac{0.885}{0.975} = \frac{1.093}{1} \text{ has been found and applied also to the parameter } e_{c0} \text{ giving } e_{c0} = 0.823 \text{ for the AP sample. This of course is a vague incorporation of the inherent fabric effect into the model but permits to have an idea of the models performance with some other interesting experiments using different sample preparations. Further modifications of the model are required to simulate the fabric effect for larger strain amplitudes.}
\]

The Poisson ratio \( v \) can be computed by direct calibration of the shear modulus \( G \) dictated by Equation 36. The factor \( r = G/K = \tilde{G}/\tilde{K} \) (Equation 35) can be used for this purpose. Figure 5(a) shows the calibration of \( v \) with an undrained triaxial test of a moist tamped (MT) sample of Toyoura sand.

To understand the calibration of the parameters \( R \), \( \beta_h \), and \( m_R \), a sketch of a typical secant shear modulus \( G_{sec} \) degradation curve with the deviator strain amplitude \( \Delta \varepsilon_s = \sqrt{2/3} \| \varepsilon^* \| \) is depicted in Figure 6(a). For very small deviator strain amplitudes \( \Delta \varepsilon_s < 10^{-5} \), the typical behavior to expect from a sand is elastic (constant \( G_{sec} \)). At larger amplitudes of approximately \( \Delta \varepsilon_s > 10^{-4} \), the plastic accumulation starts to play an important role. One can try to match such small elastic strain amplitudes with the diameter of the yield surface \( (R = \sqrt{3/2} \Delta \varepsilon_s \approx 10^{-5}) \). From the numerical point of view, this is somewhat troublesome especially by the finite element (FE) implementation. In fact, in some FE softwares, the residual tolerance for the Newton iterations are even defined in terms of very small displacement values, a fact which can affect the numerical integration when dealing with a very small yield surface \( (R \approx 10^{-5}) \). Considering this, a better approach would be to define a larger elastic domain \( R \approx 10^{-4} \) wherein the model neglects the plastic accumulation.
Figure 5. Calibration of parameters. (a) Poisson ratio $\nu$; (b) dilatancy parameter $n_d$. Experimental data from [55].

Figure 6. Secant shear modulus $G^{sec}$ degradation curve. (a) Typical curve of a sand. (b) Idealization of the degradation curve with the intergranular strain anisotropy model. The experimental curve has been sketched.

(Figure 6(b)). Actually, the value of $R \approx 10^{-4}$ has shown to be numerically stable and to deliver accurate simulations. This is probably the reason why it has been adopted as default value by the earlier intergranular strain model from Niemunis and Herle [25], also used in other models, for example, [44].

The next lines are devoted to describe the determination of parameter $\beta$. For this purpose, the strain amplitude $\| \Delta \epsilon \|$ required to reach the mobilized states $\rho = 1$, or in other words, the bounding condition $c = c_h$ is of importance. This strain amplitude will be denoted with $\| \Delta \epsilon \| = \epsilon_h$. It is reminded that at mobilized states $\rho = 1$, the constitutive model recovers the hypoplastic equation $\dot{\sigma} = \bar{E} : (\dot{\epsilon} - Y m \| \dot{\epsilon} \|$ (Equation 19), which accordingly works well for medium and large strain amplitudes $\| \Delta \epsilon \| > 10^{-3}$. Because the rate of the back-intergranular strain $\dot{\epsilon}$ decays asymptotically when applying large strains (Equation 9), it requires infinite strain to reach its bounding condition $c = c_h$. Therefore, the factor $r_h = \| \epsilon \| / (R/2) \approx 0.99$ has been introduced to indicate how close is the center of the back-intergranular strain $\epsilon$ to its bounding condition $c = c_h$. As explained before, the parameter $\beta$ controls the rate of $\epsilon$ and therefore can be calibrated according to a desired strain amplitude $\epsilon_h$. In Appendix A, it is shown that the parameter $\beta$ can be calibrated...
with the equation:

$$\beta = \left( \sqrt{6}R(\log(4) - 2\log(1 - r_h)) \right) / \left( 6\Delta\varepsilon_s - \sqrt{6}R(3 + r_h) \right)$$  \hspace{1cm} (41)$$

where $\Delta\varepsilon_s = \sqrt{\frac{2}{3}}\varepsilon_h$ is the required deviatoric strain to reach the almost bounding condition $\| \varepsilon \| = r_h R/2$. To give an example, by selecting $r_h = 0.99, R = 1.4 \times 10^{-4},$ and $\varepsilon_h = 10^{-3}$, a value of $\beta \approx 1$ is obtained. These values have been adopted to simulate the Toyoura sand; see Table I.

The shear modulus takes the residual value $G = \tilde{G}$ from Equation 36 at fully mobilized states. Within the elastic range, the shear modulus takes its maximum value $G_{\text{max}} = m_R \tilde{G}$, whereby the parameter $m_R$ can be calibrated (Figure 6(a)). Many experiments and numerical simulations have shown that setting $m_R \approx 5$ leads to accurate simulations; see, for example, the experiments given in [17, 56] or the simulations performed with some models [26, 44].

Between the elastic and mobilized states, the degradation of the secant shear modulus due to the evolution of the intergranular strain $h$ occurs (Figure 6(b)). The parameter $\chi_h$ controls an important role in this transition. It is therefore recommended to calibrate the parameter $\chi_h$ by fitting the behavior with cyclic triaxial tests. Figure 7 shows an example of a cyclic undrained triaxial test with the variation of parameter $\chi_h$. The simulations use parameters from Toyoura sand (Table I) with a medium density $e_0 = 0.8$. The strain amplitude has been chosen such that at the end of each cycle, the stiffness factor approaches to $m \approx 1$. The simulations show that with $\chi_h = 5$, the stiffness degradation and the pore water pressure accumulation are faster than with $\chi_h = 10$.

The parameter $n_d$ is used for the description of the dilatancy surface $F_d = 0$ (Equation 26). As explained by other models [39, 57], by ignoring the small elastic strains, the surface described with $F_d = 0$ would coincide with the phase of transformation line under triaxial compression tests. If this surface is described with the equation $\eta = \eta^d \approx M_e \exp(-n_d(e - e_c))$, one can solve $n_d$ from the latter equation giving $n_d = \log(M_e/\eta^d)/(e - e_c)$. Another method is to fit directly monotonic undrained triaxial for large strain amplitudes $\| \Delta\varepsilon \| > 10^{-2}$, as depicted in Figure 5(b).

5. NUMERICAL INTEGRATION

The constitutive model has been implemented in FORTRAN, following the syntax for the subroutine UMAT from the program ABAQUS standard. An explicit integration algorithm with substepping scheme has been employed. A very small subincrement size has been chosen to achieve convergence. For each subincrement, an elastic predictor with the intergranular strain evolution equation is performed to determine whether the step is in the elastic or plastic domain. The numerical integration was performed using the software INCREMENTAL DRIVER from Niemunis [58].
6. SIMULATIONS

The model performance is now evaluated by simulating some experimental results with Toyoura sand. The experiments correspond to monotonic and cyclic triaxial tests under drained and undrained conditions. The Toyoura sand is a standard sand presenting a mean particle size $D_{50} = 0.17$ mm, and maximum and minimum void ratio $e_{\max} = 0.977$ and $e_{\min} = 0.597$, respectively. The parameters

![Simulation graphs](image-url)
used by the model are listed in Table I. In all the experiments, the intergranular strain \( h \) was initialized with \( h = -R \hat{I} \) corresponding to a fully mobilized state after isotropic compression. More simulations with the ISA model can also be found in [59].

The monotonic triaxial test results reported in [55] are selected and presented in Figures 8 and 9. Actually, these set of experimental results have been used to calibrate many constitutive models [39, 40, 57] because they cover a wide range of pressures \( p \) and densities. The samples prepared with the moist tamping MT method were sheared with triaxial compression after isotropic consolidation. Drained and undrained conditions were tested. Figure 8 shows a set of undrained triaxial compression tests, and the results are grouped according to the void ratios \( e_{D} = 0.907 \), \( e_{D} = 0.833 \), and \( e_{D} = 0.735 \). The initial mean pressure ranges between \( p_{0} = 100 \) up to 3000 kPa. The simulations with the ISA model in Figure 8 show satisfactory results.

Figure 9 shows the experimental results of Toyoura sand reported in [55] of drained triaxial compression tests. The results are grouped with initial mean pressure \( p_{0} = 100 \) kPa and \( p_{0} = 500 \) kPa. The simulations capture quiet well the experiments, with some small discrepancies in the maximum deviator stress \( q_{\text{max}} \), especially for the densest sample with initial mean pressure \( p_{0} = 100 \) (Figure 9(a)).

The second set of experiments corresponds to cyclic drained triaxial tests with \( p = \text{const} \) on isotropically consolidated specimens. These tests were performed by Pradhan et al. [60] and aimed to study the stress–dilatancy relations. In contrast to the experiments of Verdugo et al. [55], these samples were prepared with the AP air pluviation method. The influence of the inherent fabric in the model has been discussed in Section 4. Figure 10 presents the result of a drained triaxial test with constant \( p = 100 \) kPa and initial void ratio \( e = 0.845 \). The results are plotted in different
spaces, including the dilatancy–stress ratio space ($q/p$ vs. $-\Delta e_v/\Delta e_3$). The complexity of this simulation lies in the fact that even after reversal loading, the dilatancy–stress ratio behavior can be interrelated with the two lines as shown in Figure 10(d). The proposed model is able to capture the stress–dilatancy behavior.

The next test was also reported by Pradhan [60] and rises the degree of challenge for the model performance evaluation: the cyclic drained triaxial test with $p = \text{const}$ is performed with ‘large’ and ‘small’ strain cycles as shown in Figure 11. The results show that after each small cycle, the material recovers the foregoing paths from the ‘large’ cycles, proving once more the existence of ‘memory’. At that point, the material follows the already known stress–dilatancy relations typical of these $p = \text{const}$ drained tests. The simulations capture most of these observations. To check the ‘memory effects’ of the model, a second simulation without the small strain cycles was also included (Figure 11(b) and (d)). Although the two simulations (with and without small cycles) are not identical during the large cycles, the response is still very similar and lacks from overshooting as shown by many other models, for example, [25, 45].

7. FINAL REMARKS

Cyclic loading is one of the most difficult challenges in the constitutive modeling of sands. All the well-known models bring advantages and shortcomings that must be very well understood. In this article, a new mathematical formulation is put into test: the yield function describing the elastic locus does not depend neither on the stress nor on the strain but rather on the intergranular strain $h$. This
is a state variable intending to describe the amount of elastic strain produced under cyclic loading and therefore allows the model to reproduce a yield surface, which can be under some conditions more realistic than other formulations. The existence of this elastic domain, even that now is defined within the intergranular strain space, can be considered as the main difference of the present model with respect to the one of Niemunis and Herle [25]. In the latter model [25], a spatial elastic domain does not exist, and therefore, the simulation of memory effects upon reloading paths is not possible.

As an additional note from the authors, it is highlighted the fact that the proposed model cannot be cataloged as an improvement of the ‘hypoplastic model with loading surface’ [39] considering that the formulations of both models are based on different concepts. Although they simulate well the memory effects upon reloading paths, only the ISA model showed plausible simulations of the stiffness increase and subsequent degradation after reversal loading.

The ISA model showed in general good simulation capabilities with the selected experiments for monotonic and cyclic loading. Many other simulations of monotonic and cyclic tests with Karlsruhe fine sand can be found in [59]. The proposed formulation allows also to build new constitutive models following similar concepts. Currently, some investigations are being made to consider the fabric effects in the mechanical behavior of sands and to propose a new model for the simulation of clays under the proposed ISA framework.

APPENDIX A: DETERMINATION OF PARAMETER $\beta$

Consider an undrained path $(h = h^*, c = c^*)$ with fully mobilized intergranular strain $\| h \| = R$ whereby a $180^\circ$ strain reversal is performed. At the beginning of the unloading path, the response is
elastic and requires a deviatoric strain $\Delta \varepsilon_s = \Delta \varepsilon^{(1)}$ equal to

$$\Delta \varepsilon^{(1)} = \sqrt{2/3} R \tag{A.1}$$

until the yield surface is reached. Subsequently, the response is plastic and the evolution equation for the intergranular strain is $\dot{\mathbf{h}} = \dot{\varepsilon} - \dot{\lambda} \mathbf{N}$. Substitution of $\dot{\varepsilon} = \dot{\mathbf{c}}$ and Equation 12 in the relation $\dot{\mathbf{h}} = \dot{\varepsilon} - \dot{\lambda} \mathbf{N}$ gives

$$\dot{\mathbf{h}} = (\dot{\varepsilon} - (\mathbf{N} : \dot{\mathbf{c}}/(1 + H_H) \mathbf{N})) = (\mathbf{N} - \mathbf{N}/(1 + H_H)) \parallel \dot{\varepsilon} \parallel \tag{A.2}$$

Equation A.2 is rewritten in terms of the deviatoric components $\dot{h}_s = \sqrt{2/3} \parallel \dot{\mathbf{h}}^* \parallel$ and $\dot{\varepsilon}_s = \sqrt{2/3} \parallel \dot{\mathbf{c}}^* \parallel$:

$$\dot{h}_s = (1 - 1/(1 + H_H)) \dot{\varepsilon}_s \tag{A.3}$$

The hardening modulus $H_H = -(\partial F_H/\partial \mathbf{c}) : \dot{\mathbf{c}}$ (Equation 12) can be found using $\partial F_H/\partial \mathbf{c} = -\mathbf{N}$ (Equation 11) and Equations 8 and 9:

$$H_H = -(\partial F_H/\partial \mathbf{c}) : \dot{\mathbf{c}} = \beta \mathbf{N} : (R/2 \mathbf{c} - \mathbf{c}) / R \tag{A.4}$$

The use of the scalar magnitude $c = \parallel \mathbf{c} \parallel (\mathbf{N} : \mathbf{c})^*$ in Equation A.4 results to

$$H_H = \beta (R/2 - c) / R \tag{A.5}$$

Substituting Equation A.5 in Equation A.3 and using the consistency condition $\dot{h}_s = \sqrt{2/3} \dot{\varepsilon}_s$ gives

$$\sqrt{2/3} \dot{\varepsilon}_s = \left(1 - \frac{R}{R + \beta (R/2 - c)}\right) \dot{\varepsilon}_s \tag{A.6}$$

According to the ISA model, the variable $c$ is bounded by $-R/2 < c < R/2$. It is desired to know the amount of deviatoric deformation $\Delta \varepsilon^{(2)}$ developed from $c = -R/2$ to $c = r_h R/2$, whereby $r_h \approx 0.99$ is a constant. This results from the integration of Equation A.6:

$$\int_{-R/2}^{r_h R/2} \left(1 - \frac{R}{R + \beta (R/2 - c)}\right)^{-1} \sqrt{2/3} \dot{\varepsilon}_s = \int_0^{\Delta \varepsilon^{(2)}} \, dc \tag{A.7}$$

Evaluation of the integral of Equation A.7 results to

$$\Delta \varepsilon^{(2)} = R (\beta + r_h \beta + \log(4) - 2 \log(1 - r_h)) / \left(\sqrt{6} \beta\right) \tag{A.8}$$

The total deviatoric strain produced upon the undrained reversal loading results from the addition of the strain increments $\Delta \varepsilon^{(1)}$ and $\Delta \varepsilon^{(2)}$ from Equations A.1 and A.8, respectively, and gives

$$\Delta \varepsilon_s = \Delta \varepsilon^{(1)} + \Delta \varepsilon^{(2)} = \sqrt{2/3} R + R (\beta + r_h \beta + \log(4) - 2 \log(1 - r_h)) / \left(\sqrt{6} \beta\right) \tag{A.9}$$

which can be solved for the parameter $\beta$ leading to the following relation:

$$\beta = \frac{\sqrt{6} R (\log(4) - 2 \log(1 - r_h))}{6 \Delta \varepsilon_s - \sqrt{6} R (3 + r_h)} \tag{A.10}$$

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REFERENCES