

On the effective stress for unsaturated soils with residual water

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In the present work an expression describing the rate of internal energy density for unsaturated soils is derived. The main difference here, with respect to preceding works, lies in the fact that the free and residual water are treated as two different phases from the soil. The latter presents intrinsic velocity equal to that of the solid phase, and is able to exchange mass with the free water phase. Under these conditions, a Bishop-type effective stress is identified as the energy-conjugated variable with the solid deformation rate, and presents a Bishop parameter equal to the effective degree of saturation. Furthermore, the negative rate of effective degree of saturation is energy-conjugated with the suction times the effective porosity. Other contributors to the internal energy come from the constituent compressibilities, the relative flow and the mass exchange between the free and residual water.

KEYWORDS: numerical modelling; partial saturation; stress analysis

INTRODUCTION

The simulation of the mechanical behaviour of unsaturated soils requires the definition of a constitutive stress $\bar{\sigma}$ able to link the solid deformation ϵ through constitutive models. The literature offers several constitutive stresses $\bar{\sigma}$, each with different definitions. Among them, the Bishop stress σ defined as (Bishop & Blight, 1963)

$$\sigma = \sigma^{\text{tot}} + p_a \mathbf{1} - \chi s \mathbf{1} \quad (1)$$

where σ^{tot} is the total stress, p_a is the pore air pressure, $s = p_a - p_w$ is the suction, p_w is the pore water pressure, $\mathbf{1}$ is the unit tensor and χ is the Bishop parameter, has become frequently used in constitutive modelling (Bolzon *et al.*, 1996; Jommi, 2000; Gallipoli *et al.*, 2003; Wheeler *et al.*, 2003; Borja, 2004; Sheng *et al.*, 2004; Tamagnini, 2004; Fuentes & Triantafyllidis, 2013). This is fairly attributed to some advantages obtained when selecting a proper function for the Bishop parameter χ (Nuth & Laloui, 2008; Gens, 2010).

Actually, the definition of the Bishop parameter χ (equation (1)) has alone emerged from different theories. Just to mention some examples, volume averaging was used by Hassanizadeh & Gray (1979) and Lewis & Schrefler (1998), entropy inequality by Hassanizadeh & Gray (1980) and Hutter *et al.* (1999), Lagrangian saturation concept by Coussy (2007) and theory of porous media with their respective balance equations by Coussy (2004), Ehlers & Ammann (2004), Borja (2006) and Borja & Koliji (2009). Detailed reviews of these approaches can be found in papers by Coussy (2007) and Gens (2010). Particularly, if the soil is idealised as a triphasic material composed by air, water and solid (without considering the phases' interfaces), and the intrinsic compressibilities of the constituents are ignored, then the Bishop stress σ can be recognised as an energy-conjugated stress with the solid deformation rate $\dot{\epsilon}$ and the corresponding Bishop parameter is equal to the degree of

saturation $\chi = S_w$ (Houlsby, 1997; Hutter *et al.*, 1999; Borja, 2006; Borja & Koliji, 2009).

The choice of $\chi = S_w$ in order to fulfil some thermodynamical considerations has been the subject of debate in recent years (Tarantino & Tombolato, 2005; Pereira & Alonso, 2009; Alonso *et al.*, 2010; Pereira *et al.*, 2010; Vlahinic *et al.*, 2011). The arguments are convincing when analysing its limitations, among them the fact that for soils presenting a residual degree of saturation $S_{w0} > 0$, the product sS_w diverges to infinite when the suction s tends to infinite as well. The direct consequence is the need to contrarrest this effect with indefinite compressive mean Bishop stress $p = (-1/3)\text{tr}\sigma \rightarrow \infty$ under constant total stress $\dot{\sigma}^{\text{tot}} = \mathbf{0}$ (see equation (1)). Considering that none of these scenarios seems realistic, many undesired restrictions are usually adopted by the modellers, such as setting $S_{w0} = 0$ in hydraulic models and choosing simultaneously appropriated hydraulic parameters in order to guarantee that the product sS_w never diverges. Others prefer to avoid large suction values in their simulations. At this point, the current authors note that the conventional formulation presents serious limitations for materials with a non-negligible amount of residual water.

To overcome these issues, some authors have proposed setting the Bishop parameter equal to the effective degree of saturation $\chi = S_w^e$ (Tarantino & Tombolato, 2005; Pereira & Alonso, 2009; Alonso *et al.*, 2010; Pereira *et al.*, 2010; Lu *et al.*, 2010) for its use in constitutive modelling; for example, see the recent models from Zhou *et al.* (2012) and Alonso *et al.* (2010). By doing this, the advantages of the Bishop stress with $\chi = S_w$ remain, and the mentioned limitations are eliminated. Above this, it is suggested to add the fact that with the choice $\chi = S_w^e$ the relations used at the saturated state to predict the critical state surface in the stress space and the hypo-elastic behaviour of the material can be used without further extensions for unsaturated states for the whole range of degree of saturation (Tarantino & Tombolato, 2005; Alonso *et al.*, 2010; Pereira *et al.*, 2010). It seems that the only controversial point is that choosing $\chi = S_w^e$ is not consistent with the energy-conjugated stress already identified in many works, whereby the corresponding Bishop parameter is equal to the degree of saturation $\chi = S_w$ (Houlsby, 1997; Borja, 2006; Coussy, 2007). To the authors' knowledge, none of these theories recognises the residual

and free water as different phases with their respective properties, and therefore it is not surprising to find the aforementioned limitations.

In the present work, an equation describing the rate of internal energy density is derived in order to identify the energy-conjugated variables for unsaturated soils with residual water. In contrast to preceding works, the residual and free water will be distinguished as different phases from the material. The theory of porous media (TPM) will be used and a similar procedure as that used by Borja (2006) will be followed. For the sake of simplicity, the internal energy produced by the interfaces, such as the contractile skin, will be ignored as in Houlsby (1997) and Borja (2006).

VARIABLES RELATED TO VOLUME FRACTIONS

For many authors the residual water is considered to be attached to the particles or trapped in the micropores (Bear, 1972; Stephens *et al.*, 1998). The direct implication is that the intrinsic velocity of the residual water is equal to that of the solid phase, whereas the remaining water fraction, called free water, is able to flow through the pores with a distinct intrinsic velocity. Without entering into explicit details of the mechanisms and interactions between these two phases, the former statement is adopted for this work as formal definitions for the residual and free water. The soil is idealised as a porous medium composed of four constituents: solid (s), free water (wf), residual water (wr) and air (a). For the sake of simplicity, it is assumed that the micropores are fully saturated with water. Mass exchange will be considered only between the free water and the residual water. The mixture has a volume $V = \sum_{\alpha} V_{\alpha}$ expressed as a sum of partial volumes V_{α} whereby $\alpha = \{s, wr, wf, a\}$ is the constituent index. Similarly, the mass of the mixture is expressed as $m = \sum_{\alpha} m_{\alpha}$. Constituents presenting different intrinsic velocities from the solid phase are denoted with the constituent index $\beta = \{wf, a\}$.

The TPM assumes a superimposed continuum in which any spatial point in the current configuration with the position vector \mathbf{x} is occupied simultaneously by points from different constituents $\mathbf{x} = \mathbf{x}_{\alpha}$. However, the constituents present different positions \mathbf{X}_{α} at the initial configuration. Therefore, the intrinsic velocity \mathbf{v}_{α} of the phase α is defined as

$$\mathbf{v}_{\alpha} = \frac{d\mathbf{x}_{\alpha}(\mathbf{X}_{\alpha}, t)}{dt}, \quad \text{whereby } \mathbf{x} = \mathbf{x}_{\alpha} = \mathbf{x}_{\alpha}(\mathbf{X}_{\alpha}, t) \quad (2)$$

where t is the current time. The kinematic relations of the TPM introduce the velocity gradient $\mathbf{L}_{\alpha} = \text{grad}\mathbf{v}_{\alpha}$ and its symmetric component $\dot{\boldsymbol{\epsilon}}_{\alpha} = (\mathbf{L}_{\alpha} + \mathbf{L}_{\alpha}^T)/2$. Notice that $\mathbf{1} : \dot{\boldsymbol{\epsilon}}_{\alpha} = \text{div}\mathbf{v}_{\alpha}$ holds.

A Lagrangian description is selected whereby the primary kinematic variable corresponds to the solid phase with intrinsic velocity $\mathbf{v} = \mathbf{v}_s$. Considering the importance of this phase, variables omitting the constituent index α imply that it is related to the solid phase (e.g. $\boldsymbol{\epsilon}_s = \boldsymbol{\epsilon}$).

The intrinsic relative velocity \mathbf{w}_{β} and partial relative velocity \mathbf{w}^{β} are defined as

$$\mathbf{w}_{\beta} = \mathbf{v}_{\beta} - \mathbf{v} \quad (3)$$

$$\mathbf{w}^{\beta} = \phi^{\beta} \mathbf{w}_{\beta} \quad (4)$$

where ϕ^{β} is the volume fraction of the phase β . The Darcy velocity \mathbf{w}^{wf} is an example from equation (4). In general, the volume fraction ϕ^{α} of the phase α is defined as

$$\phi^{\alpha} = \frac{dV_{\alpha}}{dV} \quad (5)$$

Particularly in soil mechanics and hydrogeology the total porosity is defined as $n = \phi^{wf} + \phi^{wr} + \phi^a = 1 - \phi^s$ and the effective porosity is $n^e = \phi^{wf} + \phi^a = 1 - \phi^s - \phi^{wr}$. The

time derivative operator d/dt always follows the motion of the solid phase except for the case when the constituent index α is introduced, for example, d^{α}/dt . These operators can be shown to be related to the following expression

$$\frac{d^{\alpha}\square}{dt} = \frac{d\square}{dt} + (\text{grad}\square)\mathbf{w}_{\alpha}, \quad (6)$$

where \square is an arbitrary variable. From the property $V = \sum_{\alpha} V_{\alpha}$ and equation (5) arises the saturation condition imposing the following constraint

$$\sum_{\alpha} \phi^{\alpha} = 1, \quad \text{or} \quad \sum_{\alpha} \frac{d\phi^{\alpha}}{dt} = 0 \quad (7)$$

The effective pore fraction S_{β}^e is introduced, defined as the ratio between the volume V_{β} and the volume of the pores ($= V_{wf} + V_a$), which may be also expressed as

$$S_{\beta}^e = \frac{\phi^{\beta}}{1 - \phi^s - \phi^{wr}} \quad (8)$$

Particularly the variable $S_{wf}^e = (S_w - S_{w0})/(1 - S_{w0})$ is identified as the effective degree of saturation, whereby $S_w = (\phi^{wf} + \phi^{wr})/(1 - \phi^s)$ is the degree of saturation and $S_{w0} = \phi^{wr}/(1 - \phi^s)$ is the residual degree of saturation. The saturation constraint imposes

$$\sum_{\beta} S_{\beta}^e = 1, \quad \text{or} \quad \sum_{\beta} \frac{dS_{\beta}^e}{dt} = 0 \quad (9)$$

Differentiation of equation (8) yields the following relation

$$\begin{aligned} \frac{d\phi^{\beta}}{dt} &= \frac{dS_{\beta}^e}{dt}(1 - \phi^s - \phi^{wr}) - S_{\beta}^e \left(\frac{d\phi^s}{dt} + \frac{d\phi^{wr}}{dt} \right) \\ &= \frac{dS_{\beta}^e}{dt} n^e - S_{\beta}^e \left(\frac{d\phi^s}{dt} + \frac{d\phi^{wr}}{dt} \right) \end{aligned} \quad (10)$$

It is assumed that the partial Cauchy stresses $\boldsymbol{\sigma}^{\alpha}$ for all the phases except for the solid phase are isotropic and defined as

$$\boldsymbol{\sigma}^{\alpha} = -\phi^{\alpha} p_{\alpha} \mathbf{1}, \quad \alpha = \{wr, wf, a\} \quad (11)$$

whereby p_{α} is the mean pressure of the phase $\alpha = \{wr, wf, a\}$. The intrinsic density ρ_{α} and partial density ρ^{α} are defined as

$$\rho_{\alpha} = \frac{dm_{\alpha}}{dV_{\alpha}}, \quad \rho^{\alpha} = \frac{dm_{\alpha}}{dV} \quad (12)$$

Notice that $\rho^{\alpha} = \phi^{\alpha} \rho_{\alpha}$ holds. The total density of the mixture is computed with $\rho = \sum_{\alpha} \rho^{\alpha}$. Consider now the compressible constituents with intrinsic density ρ_{α} depending on the mean pressure p_{α} , that is $\rho_{\alpha} = \rho_{\alpha}(p_{\alpha})$. The following model is considered

$$\frac{d^{\alpha} p_{\alpha}}{dt} = \frac{K_{\alpha} d^{\alpha} \rho_{\alpha}}{\rho_{\alpha} dt} \quad (13)$$

whereby K_{α} is the intrinsic bulk modulus of the phase α . Differentiation of $\rho^{\alpha} = \phi^{\alpha} \rho_{\alpha}$ and using equation (13) yields the following relation

$$\frac{d^{\alpha} \rho^{\alpha}}{dt} = \frac{d^{\alpha}}{dt} (\phi^{\alpha} \rho_{\alpha}) = \frac{\rho_{\alpha} \phi^{\alpha} d^{\alpha} p_{\alpha}}{K_{\alpha} dt} + \rho_{\alpha} \frac{d^{\alpha} \phi^{\alpha}}{dt} \quad (14)$$

The transformation rule from equation (6) together with the Darcy flow vector \mathbf{w}^{α} definition (equation (4)) allow the last equation to be expressed in terms of the solid phase derivative d/dt

$$\frac{d^{\alpha}\rho^{\alpha}}{dt} = \frac{\rho_{\alpha}\phi^{\alpha}}{K_{\alpha}} \frac{dp_{\alpha}}{dt} + \frac{\rho_{\alpha}}{K_{\alpha}} \text{grad}p_{\alpha}\mathbf{w}^{\alpha} + \rho_{\alpha} \frac{d\phi^{\alpha}}{dt} + \rho_{\alpha} \text{grad}\phi^{\alpha}\mathbf{w}_{\alpha} \quad (15)$$

For the description of the solid matrix compressibility, the following model described by Borja (2006) is used

$$\frac{dp_s}{dt} = -\frac{\bar{K}}{\phi^s} \mathbf{1} : \dot{\boldsymbol{\epsilon}} \quad (16)$$

where \bar{K} is the bulk modulus of the solid matrix.

CONSERVATION OF MASS

The localised conservation of mass for each phase under the assumptions mentioned in the previous section is described by the following equations

$$\frac{d\rho^s}{dt} + \rho^s \text{div}\mathbf{v} = 0 \quad (17)$$

$$\frac{d\rho^{wr}}{dt} + \rho^{wr} \text{div}\mathbf{v} = \hat{\rho}_{wr} \quad (18)$$

$$\frac{d^{\beta}\rho^{\beta}}{dt} + \rho^{\beta} \text{div}\mathbf{v}_{\beta} = -\hat{\rho}_{wr}\delta_{\beta,wf}, \quad \beta = \{wf, a\} \quad (19)$$

where $\hat{\rho}_{wr}$ is the rate of mass exchange between the residual and free water phase and the symbol $\delta_{\beta,wf} = 1$ when $\beta = wf$ and $\delta_{\beta,wf} = 0$ when $\beta = a$. Equations (17) and (18) are expanded by introducing the equation (14) and dividing by the intrinsic density ρ_{α} . The conservation of mass for the pore phases (free water and air) described with equation (19) is also expanded using equations (15) and (3) and dividing by ρ_a . After doing this, the following set of equations, representing the balance of mass of the phases, is obtained

$$\frac{d\phi^s}{dt} + \frac{\phi^s}{K_s} \frac{dp_s}{dt} + \phi^s \text{div}\mathbf{v} = 0 \quad (20)$$

$$\frac{d\phi^{wr}}{dt} + \frac{\phi^{wr}}{K_{wr}} \frac{dp_{wr}}{dt} + \phi^{wr} \text{div}\mathbf{v} = \frac{\hat{\rho}_{wr}}{\rho_{wr}} \quad (21)$$

$$\begin{aligned} \frac{d\phi^{\beta}}{dt} + \frac{\phi^{\beta}}{K_{\beta}} \frac{dp_{\beta}}{dt} + \phi^{\beta} \text{div}\mathbf{v} = & -\text{grad}\phi^{\beta}\mathbf{w}_{\beta} - \phi^{\beta} \text{div}\mathbf{w}_{\beta} \\ & - \frac{1}{K_{\beta}} \text{grad}p_{\beta}\mathbf{w}_{\beta} - \frac{\hat{\rho}_{wr}\delta_{\beta,wf}}{\rho_{\beta}} \end{aligned} \quad (22)$$

With the sum of equation (20) and equation (21) the conservation of mass of the residual water and the solid phase can be expressed in a single equation. Notice that this additional step was not done in the work by Borja & Koliji (2009). Doing this yields the following equation

$$\frac{d\phi^s}{dt} + \frac{d\phi^{wr}}{dt} + \frac{\phi^s}{K_s} \frac{dp_s}{dt} + \frac{\phi^{wr}}{K_{wr}} \frac{dp_{wr}}{dt} + (\phi^s + \phi^{wr}) \text{div}\mathbf{v} = \frac{\hat{\rho}_{wr}}{\rho_{wr}} \quad (23)$$

Using equation (23) in equation (10) gives

$$\begin{aligned} \frac{d\phi^{\beta}}{dt} = \frac{dS_{\beta}^e}{dt} n^e - S_{\beta}^e \left[-\frac{\phi^s}{K_s} \frac{dp_s}{dt} - \frac{\phi^{wr}}{K_{wr}} \frac{dp_{wr}}{dt} \right. \\ \left. - (\phi^s + \phi^{wr}) \text{div}\mathbf{v} + \frac{\hat{\rho}_{wr}}{\rho_{wr}} \right] \end{aligned} \quad (24)$$

Replacing equation (24) in equation (22) yields the following expression representing the balance of mass of the mixture

$$\begin{aligned} n^e \frac{dS_{\beta}^e}{dt} + S_{\beta}^e \left(\frac{\phi^s}{K_s} \frac{dp_s}{dt} + \frac{\phi^{wr}}{K_{wr}} \frac{dp_{wr}}{dt} \right) + \frac{\phi^{\beta}}{K_{\beta}} \frac{dp_{\beta}}{dt} \\ + S_{\beta}^e \text{div}\mathbf{v} = -\text{grad}\phi^{\beta}\mathbf{w}_{\beta} - \frac{1}{K_{\beta}} \text{grad}p_{\beta}\mathbf{w}_{\beta} \\ - \phi^{\beta} \text{div}\mathbf{w}_{\beta} - \hat{\rho}_{wr} \left(\frac{\delta_{\beta,wf}}{\rho_{wf}} - \frac{S_{\beta}^e}{\rho_{wr}} \right) \end{aligned} \quad (25)$$

IDENTIFICATION OF ENERGY CONJUGATED PAIRS

The rate of internal energy density of the mixture can be expressed as

$$\rho \dot{\epsilon} = \sum_{\alpha} \boldsymbol{\sigma}^{\alpha} : \dot{\boldsymbol{\epsilon}}_{\alpha} - \text{div}\mathbf{q} + \rho r \quad (26)$$

where \mathbf{q} is the heat influx vector, r is the external heat supply and ρ is the mixture density (see second section 'Variables related to volume fractions'). The equation (26) can be rewritten by using equation (11), the definition of the total stress $\boldsymbol{\sigma}^{\text{tot}} = \sum_{\alpha} \boldsymbol{\sigma}^{\alpha}$ and the identity $\mathbf{1} : (\dot{\boldsymbol{\epsilon}}_{\alpha} - \dot{\boldsymbol{\epsilon}}) = \text{div}\mathbf{w}_{\alpha}$

$$\rho \dot{\epsilon} = \boldsymbol{\sigma}^{\text{tot}} : \dot{\boldsymbol{\epsilon}} - \sum_{\beta} \phi^{\beta} p_{\beta} \text{div}\mathbf{w}_{\beta} - \text{div}\mathbf{q} + \rho r \quad (27)$$

Similar to Houlsby (1997), it is noted that equation (27) can be reorganised in several ways such that different energy-conjugated pairs can be recognised. Particularly, the following equation is obtained by using equations (16) and (25) in equation (27)

$$\begin{aligned} \rho \dot{\epsilon} = \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} + \sum_{\beta} \left[n^e \frac{dS_{\beta}^e}{dt} + S_{\beta}^e \frac{\phi^{wr}}{K_{wr}} \frac{dp_{wr}}{dt} + \frac{\phi^{\beta}}{K_{\beta}} \frac{dp_{\beta}}{dt} \right. \\ \left. + \text{grad}\phi^{\beta}\mathbf{w}_{\beta} + \frac{1}{K_{\beta}} \text{grad}p_{\beta}\mathbf{w}_{\beta} \right. \\ \left. + \hat{\rho}_{wr} \left(\frac{\delta_{\beta,wf}}{\rho_{wf}} - \frac{S_{\beta}^e}{\rho_{wr}} \right) \right] p_{\beta} - \text{div}\mathbf{q} + \rho r \end{aligned} \quad (28)$$

whereby the energy-conjugated stress $\boldsymbol{\sigma}$ with the solid deformation rate $\dot{\boldsymbol{\epsilon}}$ is identified, representing the effective stress and defined as

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\text{tot}} + \left(1 - \frac{\bar{K}}{K_s} \right) \sum_{\beta} S_{\beta}^e p_{\beta} \mathbf{1} \quad (29)$$

By using equation (9) in equation (29) gives

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\text{tot}} + B(p_a - S_{wf}^e s) \mathbf{1}, \quad \text{with } B = 1 - \bar{K}/K_s \quad (30)$$

where B is similar to the Biot parameter and $s = p_a - p_{wf}$ is the suction defined now as the difference between the pore air pressure p_a and the free water pressure p_{wf} . The effective stress $\boldsymbol{\sigma}$ obtained in equation (30) coincides with the one proposed by Lu *et al.* (2010) when the solid phase is considered as incompressible $\bar{K}/K_s = 0$ and with the one proposed by Alonso *et al.* (2010) under the assumption of saturated micropores adopted in the present work. One can show that when considering an additional phase describing the trapped air in the micropores and moving with velocity \mathbf{v} , the effective stress from Alonso *et al.* (2010) for the general case is obtained. Other similar effective stresses can be also recognised by introducing some simplifications and they are summarised in Table 1.

Using the relation $\sum_{\beta} n^e (dS_{\beta}^e/dt) p_{\beta} = -n^e s (dS_{wf}^e/dt)$ equation (28) is rewritten as follows

Table 1. Particular forms of the effective stress in equation (30) by introducing some assumptions

Effective stress	Assumptions	Similar formulations
$\sigma^{\text{tot}} + B(p_a - S_{wf}^e s) \mathbf{1}$	None	—
$\sigma^{\text{tot}} + p_a \mathbf{1} - S_{wf}^e s \mathbf{1}$	$K/K_s = 0$	Alonso <i>et al.</i> (2010), Lu <i>et al.</i> (2010)
$\sigma^{\text{tot}} + B(p_a - S_{ws}) \mathbf{1}$	$S_{w0} = 0$	Skempton (1961), Borja (2006)
$\sigma^{\text{tot}} + (p_a - S_{ws}) \mathbf{1}$	$S_{w0} = 0, K/K_s = 0$	Bishop & Blight (1963), Houlsby (1997)
$\sigma^{\text{tot}} + B p_{wf} \mathbf{1}$	$S_{wf}^e = 1$	Nur & Byerlee (1971)
$\sigma^{\text{tot}} + p_{wf} \mathbf{1}$	$S_{wf}^e = 1, K/K_s = 0$	Terzaghi (1943)

$$\begin{aligned} \rho \dot{\mathbf{e}} = & \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - n^e s \frac{dS_{wf}^e}{dt} + \sum_{\beta} \left[S_{\beta}^e \frac{\phi^{wr}}{K_{wr}} \frac{dp_{wr}}{dt} + \frac{\phi^{\beta}}{K_{\beta}} \frac{dp_{\beta}}{dt} \right. \\ & + \text{grad} \phi^{\beta} \mathbf{w}_{\beta} + \frac{1}{K_{\beta}} \text{grad} p_{\beta} \mathbf{w}^{\beta} \\ & \left. + \hat{\rho}_{wr} \left(\frac{\delta_{\beta, wf}}{\rho_{wf}} - \frac{S_{\beta}^e}{\rho_{wr}} \right) \right] p_{\beta} - \text{div} \mathbf{q} + \rho r \end{aligned} \quad (31)$$

Equation (31) reveals that an ‘effective modified’ suction equals the suction times the effective porosity ($= sn^e$) is energy-conjugated with the negative rate of effective degree of saturation $-dS_{wf}^e/dt$. Notice that the conjugated pair differs from the ones recognised by Houlsby (1997), whereby the modified suction ($= ns$) is energy-conjugated with the negative rate of degree of saturation $-dS_w/dt$. The terms $\sum_{\beta} (S_{\beta}^e \phi^{wr}/K_{wr} (dp_{wr}/dt) + \phi^{\beta}/K_{\beta} (dp_{\beta}/dt) + 1/K_{\beta} \text{grad} p_{\beta} \mathbf{w}^{\beta}) p_{\beta}$ correspond to the contribution given by the constituents’ intrinsic compression. The term $\sum_{\beta} (\text{grad} \phi^{\beta} \mathbf{w}_{\beta}) p_{\beta}$ represents the contribution from the relative flow. The term $\sum_{\beta} \hat{\rho}_{wr} (\delta_{\beta, wf}/\rho_{wf} - S_{\beta}^e/\rho_{wr})$ represents the contribution given by the mass exchange between the residual and free water. The other terms $-\text{div} \mathbf{q} + \rho r$ contribute when there exists heat flux or volumetric heat supply.

CLOSURE

The equation describing the rate of internal energy density has been revisited. In this particular work, the unsaturated soil has been idealised as a four-phasic material considering the residual and free water as two different phases. The residual water is assumed to present intrinsic velocity equal to the solid phase, and allowed to exchange mass with the free water. Intrinsic compressibilities of the phases have also been considered. Differing from most of the reported energy-conjugated stresses in the literature, it has been identified that the stress $\boldsymbol{\sigma} = \sigma^{\text{tot}} + B p_a \mathbf{1} - B \chi s \mathbf{1}$ with the Biot parameter B and Bishop parameter equal to the effective degree of saturation $\chi = S_{wf}^e$. On the other hand, the negative rate of the effective degree of saturation $-dS_{wf}^e/dt$ is conjugated with an ‘effective modified’ suction equal to the product sn^e , whereby the suction s is now defined as the difference between the air pressure p_a and the free water pressure p_{wf} , and n^e is the effective porosity. The equation also indicates that mass exchange between the free and residual water phases contributes to the rate of internal energy density. Notice that the new definition of the suction $s = p_a - p_{wf}$ coincides with the suction measured by triaxial tests and other experiments using ceramic discs. Hence, this work is only being more precise for some definitions when considering the residual water as a different phase.

As a final recommendation for modellers, special care should be taken when selecting the governing equations for the numerical implementation, for example, when using the finite-element method. Researchers using the TPM should note that

the balance of mass presented in the third section ‘Conservation of mass’ is consistent with the choice of the Bishop parameter equal to the effective degree of saturation $\chi = S_{wf}^e$.

NOTATION

$\mathbf{0}$	zero tensor (all components equal to zero)
$\mathbf{1}$	Kronecker delta tensor
B	Biot parameter
$\dot{\mathbf{e}}$	rate of internal energy of the mixture (kPa m ³ /(s kg))
\bar{K}	bulk modulus of the solid matrix (kPa)
K_{α}	intrinsic bulk modulus of the phase α (kPa)
\mathbf{L}_{α}	velocity gradient of the phase α (1/seg)
m_{α}	mass of the phase α (kg)
n	porosity
n^e	effective porosity
p	effective stress mean pressure (kPa)
p_{α}	mean pressure of the phase α (kPa)
\mathbf{q}	heat influx (kPa m/s)
r	external heat supply (kPa m ³ /(s kg))
S_w	degree of saturation
S_{w0}	residual degree of saturation
S_w^e	effective degree of saturation ($S_w^e = S_{wf}^e$)
S_{β}^e	effective degree of saturation for the phase β
s	suction (kPa)
t	time segment
V_{α}	volume of the phase α (m ³)
\mathbf{v}_{α}	intrinsic velocity of the phase α (m/s)
\mathbf{w}_{α}	intrinsic relative velocity of the phase α (m/s)
\mathbf{w}^{α}	partial relative velocity of the phase α (m/s)
\mathbf{X}_{α}	initial position vector of the phase α (m)
\mathbf{x}_{α}	current position vector of the phase α (m)
α	constituent/phase index
β	pore constituent/phase index = {wf, a}
$\delta_{\beta\alpha}$	Kronecker delta operator between the phases β and α
$\boldsymbol{\varepsilon}_{\alpha}$	strain tensor of the phase α
ρ	total density of the mixture (kg/m ³)
ρ^{α}	partial density of phase α (kg/m ³)
ρ_{α}	intrinsic density of phase α (kg/m ³)
$\hat{\rho}_{\alpha}$	rate of mass exchange of the phase α with other phases (kg/m ³ /s)
$\boldsymbol{\sigma}$	effective stress tensor (kPa)
$\boldsymbol{\sigma}$	constitutive stress (kPa)
$\boldsymbol{\sigma}^{\alpha}$	partial Cauchy stress of the phase α (kPa)
$\boldsymbol{\sigma}^{\text{tot}}$	total stress tensor (kPa)
ϕ^{α}	volume fraction of the phase α
χ	Bishop’s parameter
\sqcup	arbitrary tensorial variable

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